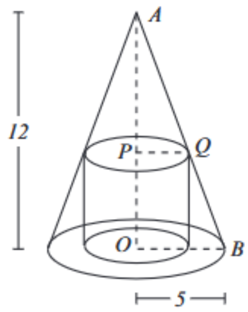
2

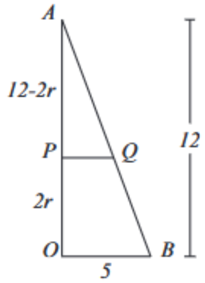
3D GEOMETRY

2001

21. **(B)** Let the cylinder have radius r and height $2r$. Since $\triangle APQ$ is similar to $\triangle AOB$, we have

$$\frac{12-2r}{r}=\frac{12}{5}, \text{ so } r=\frac{30}{11}$$





2007A

21. **Answer (C):** Since the surface area of the original cube is 24 square meters, each face of the cube has a surface area of $24/6 = 4$ square meters, and the side length of this cube is 2 meters. The sphere inscribed within the cube has diameter 2 meters, which is also the length of the diagonal of the cube inscribed in the sphere. Let l represent the side length of the inscribed cube. Applying the Pythagorean Theorem twice gives

$$l^2 + l^2 + l^2 = 2^2 = 4.$$

Hence each face has surface area

$$l^2 = \frac{4}{3} \text{ square meters.}$$

So the surface area of the inscribed cube is $6 \cdot (4/3) = 8$ square meters.

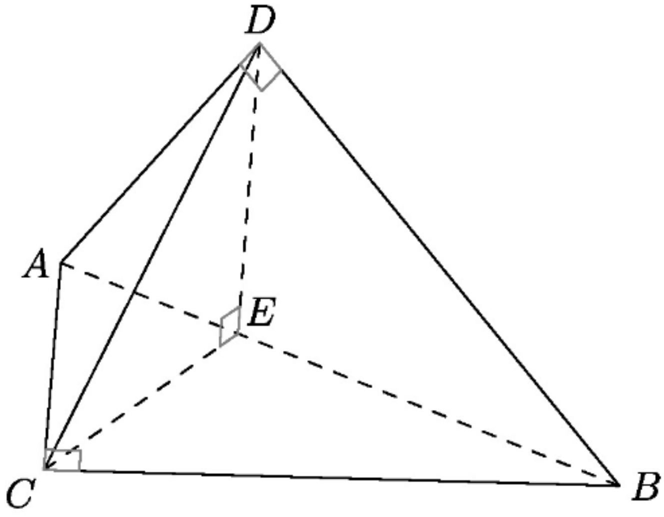
2008A

21. **Answer (A):** All sides of $ABCD$ are of equal length, so $ABCD$ is a rhombus. Its diagonals have lengths $AC = \sqrt{3}$ and $BD = \sqrt{2}$, so its area is

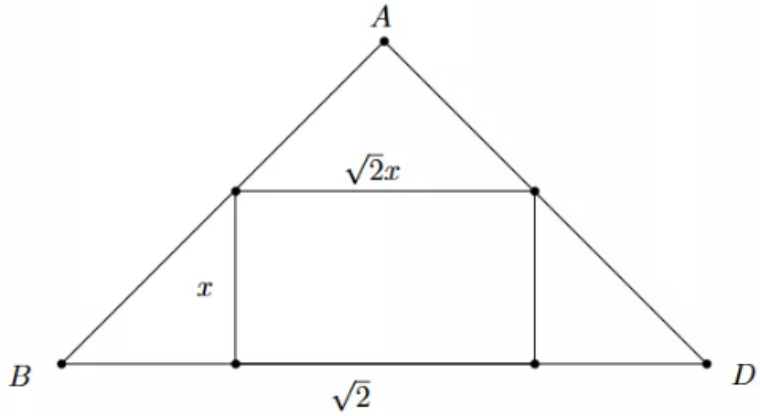
$$\frac{1}{2}\sqrt{3} \cdot \sqrt{2} = \frac{\sqrt{6}}{2}.$$

2015A

21. **Answer (C):** Triangles ABC and ABD are $3-4-5$ right triangles with area 6. Let $\overline{CE}$ be the altitude of $\triangle ABC$. Then $CE = \frac{12}{5}$. Likewise in $\triangle ABD$, $DE = \frac{12}{5}$. Triangle CDE has sides $\frac{12}{5}$, $\frac{12}{5}$, and $\frac{12}{5}\sqrt{2}$, so it is an isosceles right triangle with right angle CED. Therefore $\overline{DE}$ is the altitude of the tetrahedron to base ABC. The tetrahedron's volume is $\frac{1}{3} \cdot 6 \cdot \frac{12}{5} = \frac{24}{5}$.



2011B 22. **Answer (A):** Let A be the apex of the pyramid, and let the base be the square $BCDE$. Then $AB = AD = 1$ and $BD = \sqrt{2}$, so $\triangle BAD$ is an isosceles right triangle. Let the cube have edge length x. The intersection of the cube with the plane of $\triangle BAD$ is a rectangle with height x and width $\sqrt{2}x$. It follows that $\sqrt{2} = BD = 2x + \sqrt{2}x$, from which $x = \sqrt{2} - 1$.



Hence the cube has volume

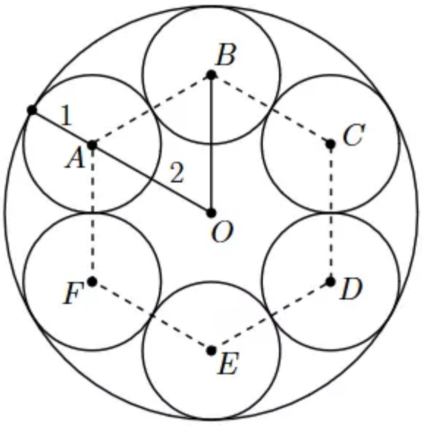
$$(\sqrt{2}-1)^3 = (\sqrt{2})^3 - 3(\sqrt{2})^2 + 3\sqrt{2} - 1 = 5\sqrt{2} - 7.$$

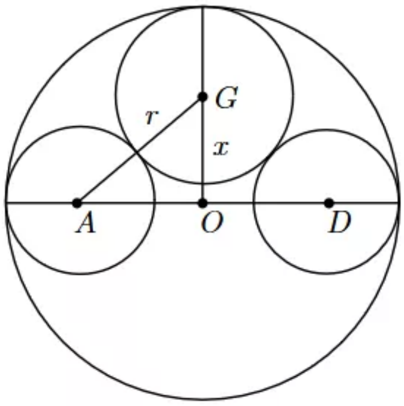
OR

Let A be the apex of the pyramid, let O be the center of the base, let P be the midpoint of one base edge, and let the cube intersect $\overline{AP}$ at Q. Let a coordinate plane intersect the pyramid so that O is the origin, A on the positive y-axis, and $P = \left(\frac{1}{2}, 0\right)$. Segment AP is an altitude of a lateral side of the pyramid, so $AP = \frac{\sqrt{3}}{2}$, and it follows that $A = \left(0, \frac{\sqrt{2}}{2}\right)$. Thus the equation of line AP is $y = \frac{\sqrt{2}}{2} - \sqrt{2}x$. If the side length of the cube is s, then $Q = \left(\frac{s}{2}, s\right)$, so $s = \frac{\sqrt{2}}{2} - \sqrt{2} \cdot \frac{s}{2}$. Solving gives $s = \sqrt{2} - 1$, and the result follows that in the first solution.

2013A

22. **Answer (B):** Let the vertices of the regular hexagon be labeled in order A, B, C, D, E, and F. Let O be the center of the hexagon, which is also the center of the largest sphere. Let the eighth sphere have center G and radius r. Because the centers of the six small spheres are each a distance 2 from O and the small spheres have radius 1, the radius of the largest sphere is 3. Because G is equidistant from A and D, the segments $\overline{GO}$ and $\overline{AO}$ are perpendicular. Let x be the distance from G to O. Then $x + r = 3$. The Pythagorean Theorem applied to $\triangle AOG$ gives $(r+1)^2 = 2^2 + x^2 = 4 + (3-r)^2$, which simplifies to $2r + 1 = 13 - 6r$, so $r = \frac{3}{2}$. Note that this shows that the eighth sphere is tangent to $\overline{AD}$ at O.





2004B 23. **(B)** If the orientation of the cube is fixed, there are $2^6 = 64$ possible arrangements of colors on the faces. There are

$$2\binom{6}{6} = 2$$

arrangements in which all six faces are the same color and

$$2\binom{6}{5} = 12$$

arrangements in which exactly five faces have the same color. In each of these cases the cube can be placed so that the four vertical faces have the same color. The only other suitable arrangements have four faces of one color, with the other color on a pair of opposing faces. Since there are three pairs of opposing faces, there are $2(3) = 6$ such arrangements. The total number of suitable arrangements is therefore $2 + 12 + 6 = 20$, and the probability is $20/64 = 5/16$.

2007B 23. **Answer (E):** Let h be the altitude of the original pyramid. Then the altitude of the smaller pyramid is $h - 2$. Because the two pyramids are similar, the ratio of their altitudes is the square root of the ratio of their surface areas. Thus $h/(h-2) = \sqrt{2}$, so

$$h = \frac{2\sqrt{2}}{\sqrt{2}-1} = 4 + 2\sqrt{2}.$$

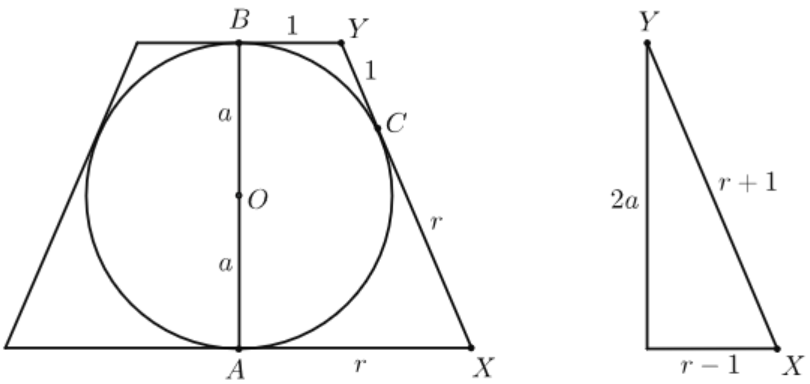
2012B 23. **Answer (B):** This situation can be modeled with a graph having these six people as vertices, in which two vertices are joined by an edge if and only if the corresponding people are internet friends. Let n be the number of friends each person has; then $1 \leq n \leq 4$. If $n = 1$, then the graph consists of three edges sharing no endpoints. There are 5 choices for Adam's friend and then 3 ways to partition the remaining 4 people into 2 pairs of friends, for a total of $5 \cdot 3 = 15$ possibilities. The case $n = 4$ is complementary, with non-friendship playing the role of friendship, so there are 15 possibilities in that case as well.

For $n = 2$, the graph must consist of cycles, and the only two choices are two triangles (3-cycles) and a hexagon (6-cycle). In the former case, there are $\binom{5}{2} = 10$ ways to choose two friends for Adam and that choice uniquely determines the triangles. In the latter case, every permutation of the six vertices determines a hexagon, but each hexagon is counted $6 \cdot 2 = 12$ times, because the hexagon can start at any vertex and be traversed in either direction. This gives $\frac{6!}{12} = 60$ hexagons, for a total of $10 + 60 = 70$ possibilities. The complementary case $n = 3$ provides 70 more. The total is therefore $15 + 15 + 70 + 70 = 170$.

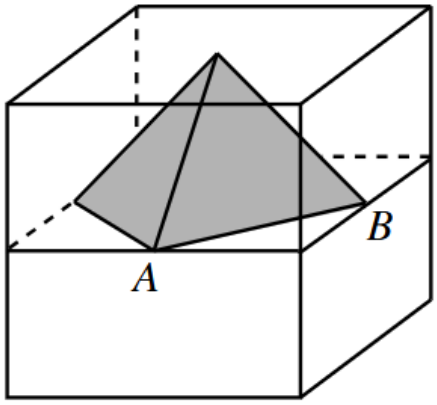
2014B 23. **Answer (E):** Assume without loss of generality that the radius of the top base of the truncated cone (frustum) is 1. Denote the radius of the bottom base by r and the radius of the sphere by a. The figure on the left is a side view of the frustum. Applying the Pythagorean Theorem to the triangle on the right yields $r = a^2$. The volume of the frustum is

$$\frac{1}{3}\pi(r^2 + r \cdot 1 + 1^2) \cdot 2a = \frac{1}{3}\pi(a^4 + a^2 + 1) \cdot 2a.$$

Setting this equal to twice the volume of the sphere, $\frac{4}{3}\pi a^3$, and simplifying gives $a^4 - 3a^2 + 1 = 0$, or $r^2 - 3r + 1 = 0$. Therefore $r = \frac{3+\sqrt{5}}{2}$.



2006A 24. **(B)** Two pyramids with square bases form the octahedron. The upper pyramid is shown.



Since the length of $\overline{AB}$ is $\sqrt{2}/2$, the base area of the pyramid is $(\sqrt{2}/2)^2 = 1/2$. The altitude of the pyramid is $1/2$, so its volume is

$$\frac{1}{3} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{12}.$$

The volume of the octahedron is $2(1/12) = 1/6$.

2011A 24. **Answer (D):** Let the tetrahedra be T_1 and T_2, and let R be their intersection. Let squares $ABCD$ and $EFGH$, respectively, be the top and bottom faces of the unit cube, with E directly under A and F directly under B. Without loss of generality, T_1 has vertices A, C, F, and H, and T_2 has vertices B, D, E, and G. One face of T_1 is $\triangle ACH$, which intersects edges of T_2 at the midpoints J, K, and L of $\overline{AC}$, $\overline{CH}$, and $\overline{HA}$, respectively. Let S be the tetrahedron with vertices J, K, L, and D. Then S is similar to T_2 and is contained in T_2, but not in R. The other three faces of T_1 each cut off from T_2 a tetrahedron congruent to S. Therefore the volume of R is equal to the volume of T_2 minus four times the volume of S.

A regular tetrahedron of edge length s has base area $\frac{\sqrt{3}}{4}s^2$ and altitude $\frac{\sqrt{6}}{3}s$, so its volume is $\frac{1}{3}\left(\frac{\sqrt{3}}{4}s^2\right)\left(\frac{\sqrt{6}}{3}s\right) = \frac{\sqrt{2}}{12}s^3$. Because the edges of tetrahedron T_2 are face diagonals of the cube, T_2 has edge length $\sqrt{2}$. Because J and K are centers of adjacent faces of the cube, tetrahedron S has edge length $\frac{\sqrt{2}}{2}$. Thus the volume of R is

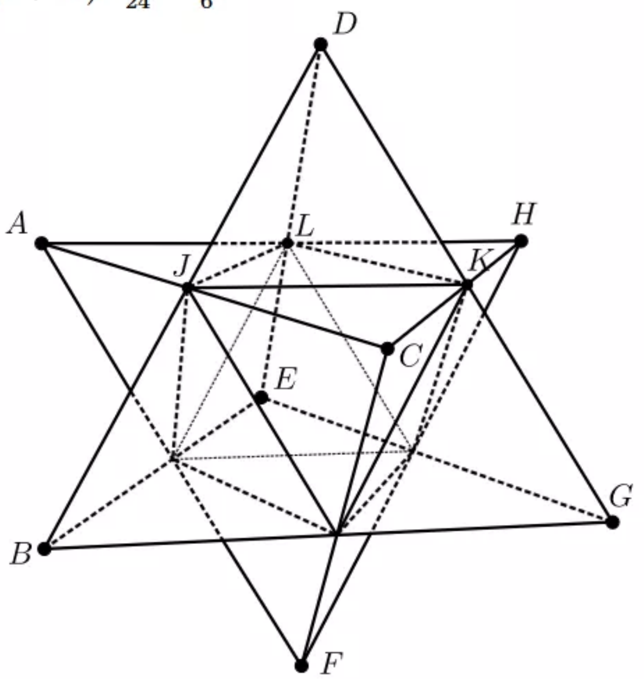
$$\frac{\sqrt{2}}{12}\left((\sqrt{2})^3 - 4\left(\frac{\sqrt{2}}{2}\right)^3\right) = \frac{1}{6}.$$

OR

Let T_1 and T_2 be labeled as in the previous solution. The cube is partitioned by T_1 and T_2 into 8 tetrahedra congruent to $DJKL$ (one for every vertex of the cube), 12 tetrahedra congruent to $AJLD$ (one for every edge of the cube), and the solid $T_1 \cap T_2$. Because the bases AJL and JLK are equilateral triangles with the same area, and the altitudes to vertex D of the tetrahedra $AJLD$ and $DJKL$ are the same, it follows that the volumes of $AJLD$ and $DJKL$ are equal. Moreover,

$$\text{Volume}(AJLD) = \frac{1}{3}\text{Area}(ALD) \cdot h_J,$$

where $h_J = \frac{1}{2}$ is the distance from J to the face ALD, and $\text{Area}(ALD) = \frac{1}{4}$. Therefore $\text{Volume}(AJLD) = \frac{1}{3} \cdot \frac{1}{4} \cdot \frac{1}{2} = \frac{1}{24}$, and thus the volume of $T_1 \cap T_2$ is equal to $1 - (8+12) \cdot \frac{1}{24} = \frac{1}{6}$.

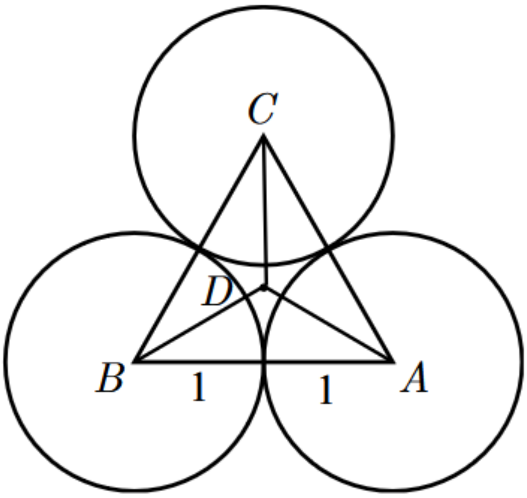


2004A 25. **(B)** Let A, B, C and E be the centers of the three small spheres and the large sphere, respectively. Then $\triangle ABC$ is equilateral with side length 2. If D is the intersection of the medians of $\triangle ABC$, then E is directly above D. Because $AE = 3$ and $AD = 2\sqrt{3}/3$, it follows that

$$DE = \sqrt{3^2 - \left(\frac{2\sqrt{3}}{3}\right)^2} = \frac{\sqrt{69}}{3}.$$

Because D is 1 unit above the plane and the top of the larger sphere is 2 units above E, the distance from the plane to the top of the larger sphere is

$$3 + \frac{\sqrt{69}}{3}.$$



2009B

25. **Answer (B):** The stripe on each face of the cube will be oriented in one of two possible directions, so there are $2^6 = 64$ possible stripe combinations on the cube. There are 3 pairs of parallel faces so, if there is an encircling stripe, then the pair of faces that do not contribute uniquely determine the stripe orientation for the remaining faces. In addition, the stripe on each face that does not contribute may be oriented in 2 different ways. Thus a total of $3 \cdot 2 \cdot 2 = 12$ stripe combinations on the cube result in a continuous stripe around the cube, and the requested probability is $\frac{12}{64} = \frac{3}{16}$.

OR

Without loss of generality, orient the cube so that the stripe on the top face goes from front to back. There are two mutually exclusive ways for there to be an encircling stripe: either the front, bottom, and back faces are painted to complete an encircling stripe with the top face's stripe, or the front, right, back, and left faces are painted to form an encircling stripe. The probability of the first cases is $(\frac{1}{2})^3 = \frac{1}{8}$, and the probability of the second case is $(\frac{1}{2})^4 = \frac{1}{16}$, so the answer is $\frac{1}{8} + \frac{1}{16} = \frac{3}{16}$.

OR

There are three possible orientations of an encircling stripe. For any one of these to appear, the four faces through which the stripe is to pass must be properly aligned. The probability of one such stripe alignment is $\left(\frac{1}{2}\right)^4 = \frac{1}{16}$. Because there are 3 such possibilities, and these events are disjoint, the total probability is $3\left(\frac{1}{16}\right) = \frac{3}{16}$.