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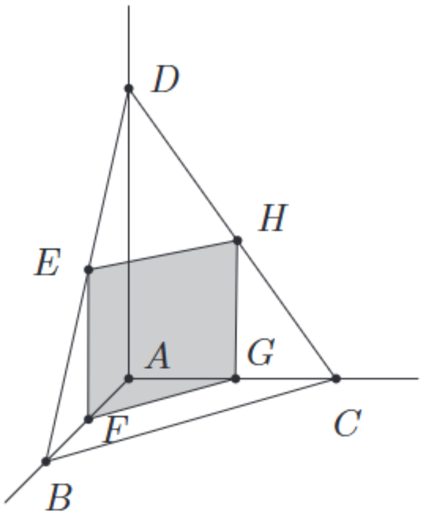
2D GEOMETRY NEED FORMULA

2012A 21. **Answer (C):** The midpoint formula gives $E = (\frac{1}{2}, 0, \frac{3}{2})$, $F = (\frac{1}{2}, 0, 0)$, $G = (0, 1, 0)$, and $H = (0, 1, \frac{3}{2})$. Note that $EF = GH = \frac{3}{2}$, $\overline{EF} \perp \overline{EH}$, $\overline{GF} \perp \overline{GH}$, and

$$EH = FG = \sqrt{\left(\frac{1}{2}\right)^2 + 1^2} = \frac{\sqrt{5}}{2}.$$

Therefore $EFGH$ is a rectangle with area $\frac{3}{2} \cdot \frac{\sqrt{5}}{2} = \frac{3\sqrt{5}}{4}$.

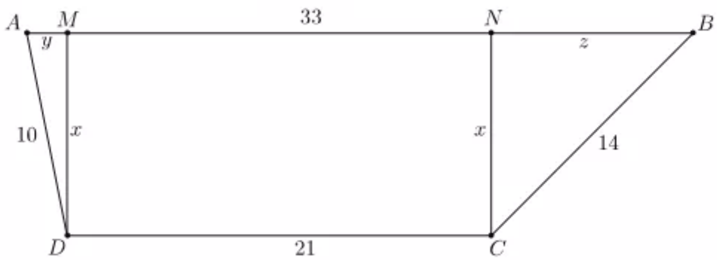


2014A

21. **Answer (E):** Setting $y = 0$ in both equations and solving for x gives $x = -\frac{5}{a} = -\frac{b}{3}$, so $ab = 15$. Only four pairs of positive integers (a, b) have product 15, namely $(1, 15)$, $(15, 1)$, $(3, 5)$, and $(5, 3)$. Therefore the four possible points on the x-axis have coordinates -5, $-\frac{1}{3}$, $-\frac{5}{3}$, and -1, the sum of which is -8.

2014B

21. **Answer (B):** Assume without loss of generality that $DA = 10$ and $BC = 14$. Let M and N be the feet of the perpendicular segments to $\overline{AB}$ from D and C, respectively. The four points A, M, N, B appear on $\overline{AB}$ in that order. Let $x = DM = CN$, $y = AM$, and $z = NB$. Then $x^2 + y^2 = 10^2 = 100$, $x^2 + z^2 = 14^2 = 196$, and $y + 21 + z = 33$. Therefore $z = 12 - y$, and it follows that $\sqrt{196 - x^2} = 12 - \sqrt{100 - x^2}$. Squaring and simplifying gives $24\sqrt{100 - x^2} = 48$, so $x^2 = 96$ and $y = \sqrt{100 - 96} = 2$. The square of the length of the shorter diagonal, $\overline{AC}$, is $(y + 21)^2 + x^2 = 23^2 + 96 = 625$, so $AC = 25$.

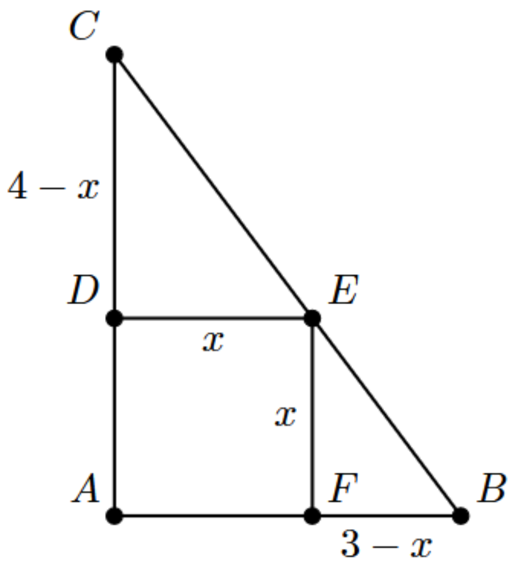


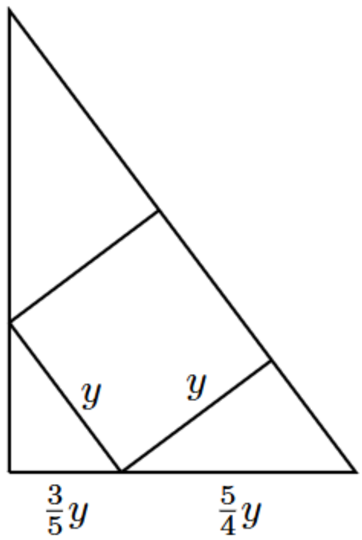
2016A 18. **Answer (C):** The sum of the four numbers on the vertices of each face must be $\frac{1}{6} \cdot 3 \cdot (1+2+\cdots+8) = 18$. The only sets of four of the numbers that include 1 and have a sum of 18 are $\{1,2,7,8\}$, $\{1,3,6,8\}$, $\{1,4,5,8\}$, and $\{1,4,6,7\}$. Three of these sets contain both 1 and 8. Because two specific vertices can belong to at most two faces, the vertices of one face must be labeled with the numbers $1,4,6,7$, and two of the faces must include vertices labeled 1 and 8. Thus 1 and 8 must mark two adjacent vertices. The cube can be rotated so that the vertex labeled 1 is at the lower left front, and the vertex labeled 8 is at the lower right front. The numbers 4, 6, and 7 must label vertices on the left face. There are $3! = 6$ ways to assign these three labels to the three remaining vertices of the left face. Then the numbers 5, 3, and 2 must label the vertices of the right face adjacent to the vertices labeled 4, 6, and 7, respectively. Hence there are 6 possible arrangements.

2016B 21. **Answer (B):** The graph of the equation is symmetric about both axes. In the first quadrant, the equation is equivalent to $x^2+y^2-x-y=0$. Completing the square gives $(x-\frac{1}{2})^2+(y-\frac{1}{2})^2=\frac{1}{2}$, so the graph in the first quadrant is an arc of the circle that is centered at $C(\frac{1}{2},\frac{1}{2})$ and contains the points $A(1,0)$ and $B(0,1)$. Because C is the midpoint of $\overline{AB}$, the arc is a semicircle. The region enclosed by the graph in the first quadrant is the union of isosceles right triangle AOB, where $O(0,0)$ is the origin, and a semicircle with diameter $\overline{AB}$. The triangle and the semicircle have areas $\frac{1}{2}$ and $\frac{1}{2}\cdot\pi\left(\frac{\sqrt{2}}{2}\right)^2=\frac{\pi}{4}$, respectively, so the area of the region enclosed by the graph in all quadrants is $4(\frac{1}{2}+\frac{\pi}{4})=\pi+2$.

2017A

21. **Answer (D):** In the first figure $\triangle FEB \sim \triangle DCE$, so $\frac{x}{3-x} = \frac{4-x}{x}$ and $x = \frac{12}{7}$. In the second figure, the small triangles are similar to the large one, so the lengths of the portions of the side of length 3 are as shown. Solving $\frac{3}{5}y + \frac{5}{4}y = 3$ yields $y = \frac{60}{37}$. Thus $\frac{x}{y} = \frac{12}{7} \cdot \frac{37}{60} = \frac{37}{35}$.



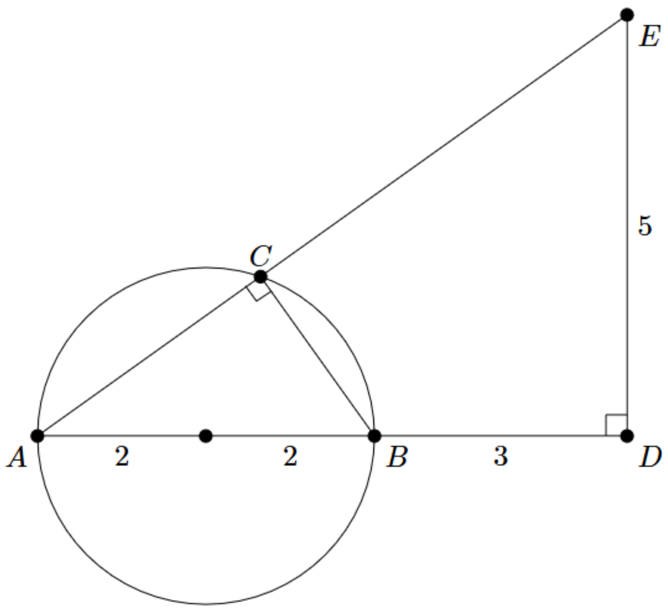


2017B

22. **Answer (D):** Because $\angle ACB$ is inscribed in a semicircle, it is a right angle. Therefore $\triangle ABC$ is similar to $\triangle AED$, so their areas are related as AB^2 is to AE^2. Because $AB^2 = 4^2 = 16$ and, by the Pythagorean Theorem,

$$AE^2 = (4+3)^2 + 5^2 = 74,$$

this ratio is $\frac{16}{74} = \frac{8}{37}$. The area of $\triangle AED$ is $\frac{35}{2}$, so the area of $\triangle ABC$ is $\frac{35}{2} \cdot \frac{8}{37} = \frac{140}{37}$.



2005B 23. **(C)** First note that $FE = (AB + DC)/2$. Because trapezoids $ABEF$ and $FECD$ have the same height, the ratio of their areas is equal to the ratio of the averages of their parallel sides. Since

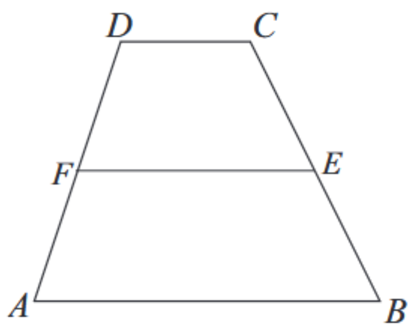
$$AB + \frac{AB + DC}{2} = \frac{3AB + DC}{2}$$

and

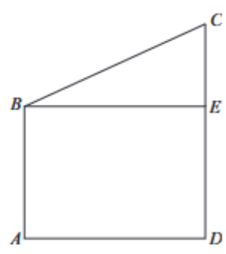
$$\frac{AB + DC}{2} + DC = \frac{AB + 3DC}{2},$$

we have

$$3AB + DC = 2(AB + 3DC) = 2AB + 6DC, \quad \text{and} \quad \frac{AB}{DC} = 5.$$



2001 24. **(B)** Let E be the foot of the perpendicular from B to $\overline{CD}$. Then $AB = DE$ and $BE = AD = 7$. By the Pythagorean Theorem,

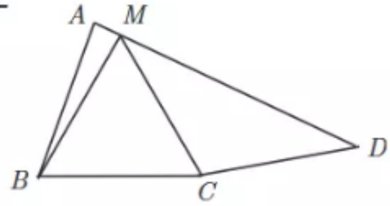


$$\begin{aligned} AD^2 = BE^2 = BC^2 - CE^2 &= (CD + AB)^2 - (CD - AB)^2 \\ &= (CD + AB + CD - AB)(CD + AB - CD + AB) \\ &= 4 \cdot CD \cdot AB. \end{aligned}$$

Hence, $AB \cdot CD = AD^2/4 = 7^2/4 = 49/4 = 12.25$.

2008B

24. **Answer (C):** Let M be on the same side of line BC as A, such that $\triangle BMC$ is equilateral. Then $\triangle ABM$ and $\triangle MCD$ are isosceles with $\angle ABM = 10^\circ$ and $\angle MCD = 110^\circ$. Hence $\angle AMB = 85^\circ$ and $\angle CMD = 35^\circ$. Therefore



$$\begin{aligned}\angle AMD &= 360^\circ - \angle AMB - \angle BMC - \angle CMD \\ &= 360^\circ - 85^\circ - 60^\circ - 35^\circ = 180^\circ.\end{aligned}$$

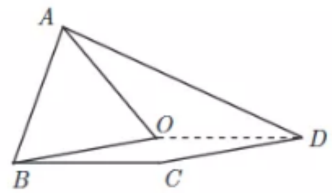
It follows that M lies on $\overline{AD}$ and $\angle BAD = \angle BAM = 85^\circ$.

OR

Let $\triangle ABO$ be equilateral as shown.

Then

$$\angle OBC = \angle ABC - \angle ABO = 70^\circ - 60^\circ = 10^\circ.$$



Because $\angle BCD = 170^\circ$ and $OB = BC = CD$, the quadrilateral $BCDO$ is a parallelogram. Thus

$OD = BC = AO$ and $\triangle AOD$ is isosceles. Let $\alpha = \angle ODA = \angle OAD$. The sum of the interior angles of $ABCD$ is 360°, so we have

$$360 = (\alpha + 60) + 70 + 170 + (\alpha + 10) \quad \text{and} \quad \alpha = 25.$$

Thus $\angle DAB = 60 + \alpha = 85^\circ$.

2011B

24. **Answer (B):** For $0 < x \le 100$, the nearest lattice point directly above the line $y = \frac{1}{2}x + 2$ is $\left(x, \frac{1}{2}x + 3\right)$ if x is even and $\left(x, \frac{1}{2}x + \frac{5}{2}\right)$ if x is odd. The slope of the line that contains this point and $(0, 2)$ is $\frac{1}{2} + \frac{1}{x}$ if x is even and $\frac{1}{2} + \frac{1}{2x}$ if x is odd. The minimum value of the slope is $\frac{51}{100}$ if x is even and $\frac{50}{99}$ if x is odd. Therefore the line $y = mx + 2$ contains no lattice point with $0 < x \le 100$ for $\frac{1}{2} < m < \frac{50}{99}$.

2016A

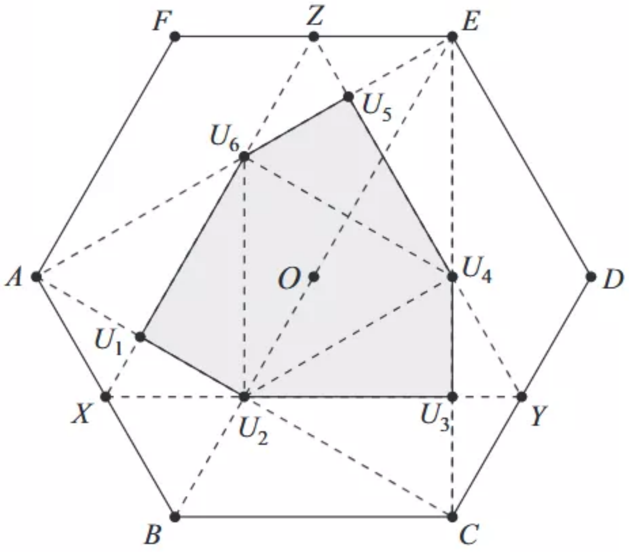
20. **Answer (B):** If a term contains all four variables a, b, c, and d, then it has the form $a^{i+1}b^{j+1}c^{k+1}d^{l+1}1^{m}$ for some nonnegative integers i, j, k, l, and m such that $(i+1)+(j+1)+(k+1)+(l+1)+m=N$ or $i+j+k+l+m=N-4$. The number of terms can be counted using the stars and bars technique. The number of linear arrangements of $N-4$ stars and 4 bars corresponds to the number of possible values of i, j, k, l, and m. Namely, in each arrangement the bars separate the stars into five groups (some of them can be empty) whose sizes are the values of i, j, k, l, and m. There are

$$\binom{N-4+4}{4}=\binom{N}{4}=\frac{N(N-1)(N-2)(N-3)}{4\cdot 3\cdot 2\cdot 1}=1001=7\cdot 11\cdot 13$$

such arrangements. So $N(N-1)(N-2)(N-3)=4\cdot 3\cdot 2\cdot 7\cdot 11\cdot 13=14\cdot 13\cdot 12\cdot 11$. Thus the answer is $N=14$.

2018B

24. **Answer (C):** Let O be the center of the regular hexagon. Points B, O, E are collinear and $BE = BO + OE = 2$. Trapezoid $FABE$ is isosceles, and $\overline{XZ}$ is its midline. Hence $XZ = \frac{3}{2}$ and analogously $XY = ZY = \frac{3}{2}$.



Denote by U_1 the intersection of $\overline{AC}$ and $\overline{XZ}$ and by U_2 the intersection of $\overline{AC}$ and $\overline{XY}$. It is easy to see that $\triangle AXU_1$ and $\triangle U_2XU_1$ are congruent $30-60-90°$ right triangles.

By symmetry the area of the convex hexagon enclosed by the intersection of $\triangle ACE$ and $\triangle XYZ$, shaded in the figure, is equal to the area of $\triangle XYZ$ minus 3 times the area of $\triangle U_2XU_1$. The hypotenuse

of $\triangle U_2XU_1$ is $XU_2 = AX = \frac{1}{2}$, so the area of $\triangle U_2XU_1$ is

$$\frac{1}{2} \cdot \frac{\sqrt{3}}{4} \cdot \left(\frac{1}{2}\right)^2 = \frac{1}{32}\sqrt{3}.$$

The area of the equilateral triangle XYZ with side length $\frac{3}{2}$ is equal to $\frac{1}{4}\sqrt{3} \cdot \left(\frac{3}{2}\right)^2 = \frac{9}{16}\sqrt{3}$. Hence the area of the shaded hexagon is

$$\frac{9}{16}\sqrt{3} - 3 \cdot \frac{1}{32}\sqrt{3} = 3\sqrt{3}\left(\frac{3}{16} - \frac{1}{32}\right) = \frac{15}{32}\sqrt{3}.$$

OR

Let U_1 and U_2 be as above, and continue labeling the vertices of the shaded hexagon counterclockwise with U_3, U_4, U_5, and U_6 as shown. The area of $\triangle ACE$ is half the area of hexagon $ABCDEF$. Triangle $U_2U_4U_6$ is the midpoint triangle of $\triangle ACE$, so its area is $\frac{1}{4}$ of the area of $\triangle ACE$, and thus $\frac{1}{8}$ of the area of $ABCDEF$. Each of $\triangle U_2U_3U_4$, $\triangle U_4U_5U_6$, and $\triangle U_6U_1U_2$ is congruent to half of $\triangle U_2U_4U_6$, so the total shaded area is $\frac{5}{2}$ times the area of $\triangle U_2U_4U_6$ and therefore $\frac{5}{2} \cdot \frac{1}{8} = \frac{5}{16}$ of the area of $ABCDEF$. The area of $ABCDEF$ is $6 \cdot \frac{\sqrt{3}}{4} \cdot 1^2$, so the requested area is $\frac{15}{32}\sqrt{3}$.