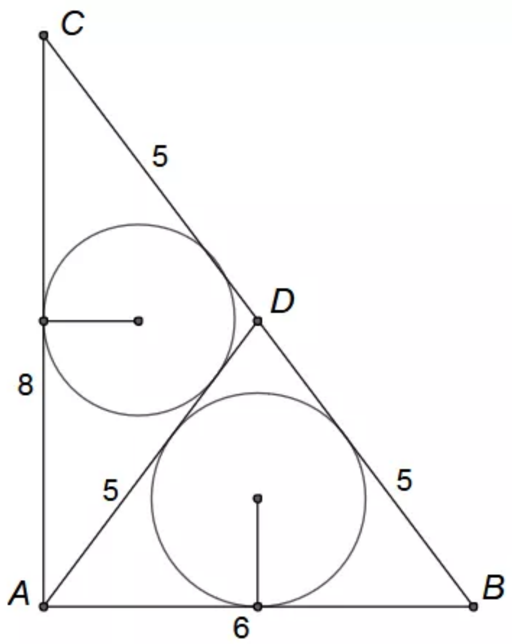
4

TRIANGLES

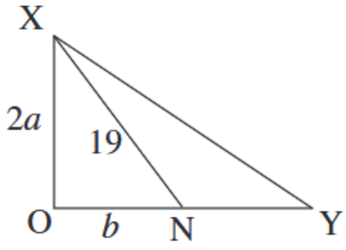
2017B 21. **Answer (D):** By the converse of the Pythagorean Theorem, $\angle BAC$ is a right angle, so $BD = CD = AD = 5$, and the area of each of the small triangles is 12 (half the area of $\triangle ABC$). The area of $\triangle ABD$ is equal to its semiperimeter, $\frac{1}{2} \cdot (5 + 5 + 6) = 8$, multiplied by the radius of the inscribed circle, so the radius is $\frac{12}{8} = \frac{3}{2}$. Similarly, the radius of the inscribed circle of $\triangle ACD$ is $\frac{4}{3}$. The requested sum is $\frac{3}{2} + \frac{4}{3} = \frac{17}{6}$.

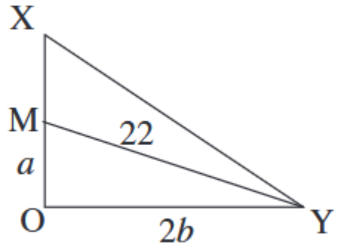


2002B

22. **(B)** Let $OM = a$ and $ON = b$. Then

$$19^2 = (2a)^2 + b^2 \quad \text{and} \quad 22^2 = a^2 + (2b)^2.$$





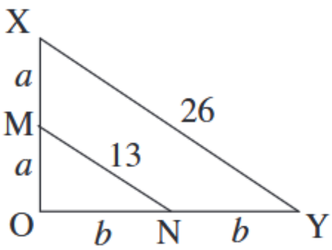
Hence

$$5(a^2 + b^2) = 19^2 + 22^2 = 845.$$

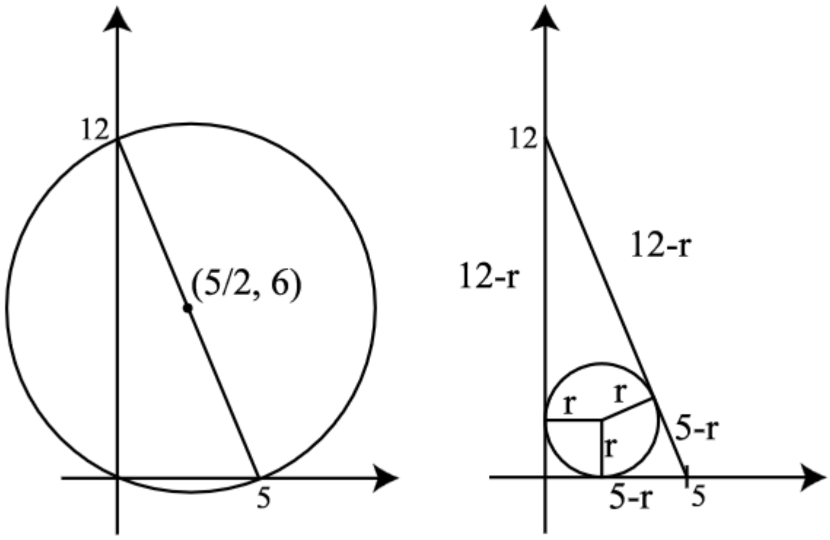
It follows that

$$MN = \sqrt{a^2 + b^2} = \sqrt{169} = 13.$$

Since $\triangle XOY$ is similar to $\triangle MON$ and $XO = 2 \cdot MO$, we have $XY = 2 \cdot MN = 26$.



2004B 22. **(D)** The triangle is a right triangle that can be placed in a coordinate system with vertices at $(0,0)$, $(5,0)$, and $(0,12)$. The center of the circumscribed circle is the midpoint of the hypotenuse, which is $(5/2, 6)$.



To determine the radius r of the inscribed circle notice that the hypotenuse of the triangle is

$$(12 - r) + (5 - r) = 13 \quad \text{so} \quad r = 2.$$

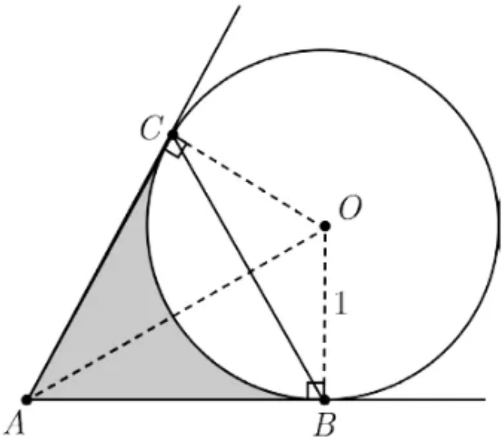
So the center of the inscribed circle is $(2, 2)$, and the distance between the two centers is

$$\sqrt{\left(\frac{5}{2} - 2\right)^2 + (6 - 2)^2} = \frac{\sqrt{65}}{2}.$$

2017A

22. **Answer (E):** Let O be the center of the circle, and without loss of generality, assume that radius $OB = 1$. Because $\triangle ABO$ is a $30-60-90°$ right triangle, $AO = 2$ and $AB = BC = \sqrt{3}$. Kite $ABOC$ has diagonals of lengths 2 and $\sqrt{3}$, so its area is $\sqrt{3}$. Because $\angle BOC = 120°$, the area of the sector cut off by $\angle BOC$ is $\frac{1}{3}\pi$. The area of the portion of $\triangle ABC$ lying outside the circle (shaded in the figure) is therefore $\sqrt{3}-\frac{1}{3}\pi$. The area of $\triangle ABC$ is $\frac{1}{4}\sqrt{3}(\sqrt{3})^2 = \frac{3}{4}\sqrt{3}$, so the requested fraction is

$$\frac{\sqrt{3}-\frac{1}{3}\pi}{\frac{3}{4}\sqrt{3}} = \frac{4}{3} - \frac{4\sqrt{3}\pi}{27}.$$

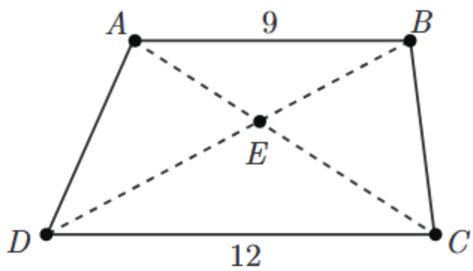


2009A

23. **Answer (E):** Because $\triangle AED$ and $\triangle BEC$ have equal areas, so do $\triangle ACD$ and $\triangle BCD$. Side $\overline{CD}$ is common to $\triangle ACD$ and $\triangle BCD$, so the altitudes from A and B to $\overline{CD}$ have the same length. Thus $\overline{AB} \parallel \overline{CD}$, so $\triangle ABE$ is similar to $\triangle CDE$ with similarity ratio

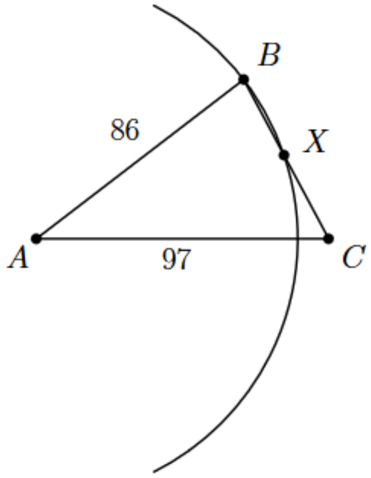
$$\frac{AE}{EC} = \frac{AB}{CD} = \frac{9}{12} = \frac{3}{4}.$$

Let $AE = 3x$ and $EC = 4x$. Then $7x = AE + EC = AC = 14$, so $x = 2$, and $AE = 3x = 6$.



2013A

23. **Answer (D):** By the Power of a Point Theorem, $BC \cdot CX = AC^2 - r^2$ where $r = AB$ is the radius of the circle. Thus $BC \cdot CX = 97^2 - 86^2 = 2013$. Since $BC = BX + CX$ and CX are both integers, they are complementary factors of 2013. Note that $2013 = 3 \cdot 11 \cdot 61$, and $CX < BC < AB + AC = 183$. Thus the only possibility is $CX = 33$ and $BC = 61$.



2013B

23. **Answer (B):** The Pythagorean Theorem applied to right triangles ABD and ACD gives $AB^2 - BD^2 = AD^2 = AC^2 - CD^2$; that is, $13^2 - BD^2 = 15^2 - (14 - BD)^2$, from which it follows that $BD = 5$, $CD = 9$, and $AD = 12$. Because triangles AED and ADC are similar,

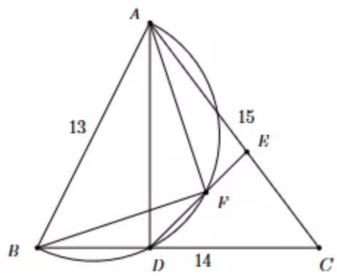
$$\frac{AE}{12} = \frac{DE}{9} = \frac{12}{15},$$

implying that $ED = \frac{36}{5}$ and $AE = \frac{48}{5}$.

Because $\angle AFB = \angle ADB = 90°$, it follows that $ABDF$ is cyclic. Thus $\angle ABD + \angle AFD = 180°$ from which $\angle ABD = \angle AFE$. Therefore right triangles ABD and AFE are similar. Hence

$$\frac{FE}{5} = \frac{\frac{48}{5}}{12},$$

from which it follows that $FE = 4$. Consequently $DF = DE - FE = \frac{36}{5} - 4 = \frac{16}{5}$.

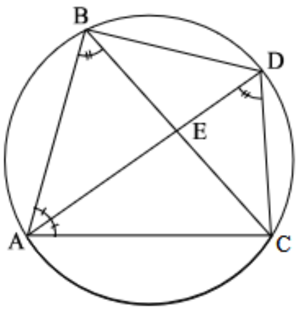


2017A

23. **Answer (B):** There are $\binom{25}{3} = \frac{25\cdot 24\cdot 23}{6} = 2300$ ways to choose three vertices, but in some cases they will fall on a line. There are $5\cdot\binom{5}{3} = 50$ that fall on a horizontal line, another 50 that fall on a vertical line, $\binom{5}{3} + 2\binom{4}{3} + 2\binom{3}{3} = 20$ that fall on a line with slope 1, another 20 that fall on a line with slope -1, and 3 each that fall on lines with slopes 2, -2, $\frac{1}{2}$, and $-\frac{1}{2}$. Therefore the answer is $2300-50-50-20-20-12 = 2148$.

2004B

24. **(B)** Suppose that AD and BC intersect at E.



Since $\angle ADC$ and $\angle ABC$ cut the same arc of the circumscribed circle, the Inscribed Angle Theorem implies that

$$\angle ABC = \angle ADC.$$

Also, $\angle EAB = \angle CAD$, so $\triangle ABE$ is similar to $\triangle ADC$, and

$$\frac{AD}{CD} = \frac{AB}{BE}.$$

By the Angle Bisector Theorem,

$$\frac{BE}{EC} = \frac{AB}{AC},$$

so

$$BE = \frac{AB}{AC}\cdot EC = \frac{AB}{AC}(BC - BE) \quad \text{and} \quad BE = \frac{AB\cdot BC}{AB+AC}.$$

Hence

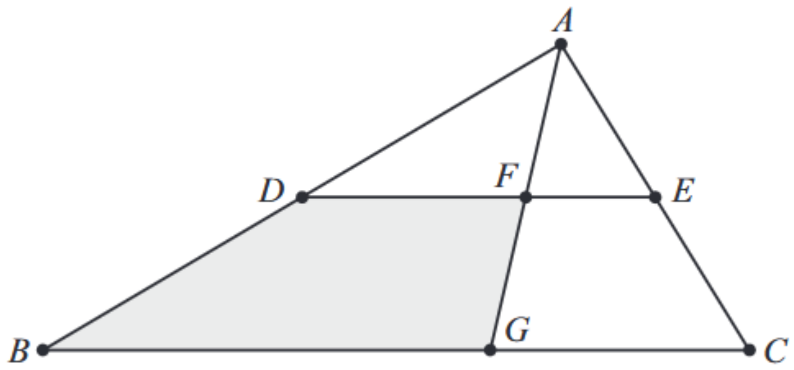
$$\frac{AD}{CD} = \frac{AB}{BE} = \frac{AB+AC}{BC} = \frac{7+8}{9} = \frac{5}{3}.$$

2017B

24. **Answer (C):** Assume without loss of generality that two of the vertices of the triangle are on the branch of the hyperbola in the first quadrant. This forces the centroid of the triangle to be the vertex $(1,1)$ of the hyperbola. Because the vertices of the triangle are equidistant from the centroid, the first-quadrant vertices must be $\left(a, \frac{1}{a}\right)$ and $\left(\frac{1}{a}, a\right)$ for some positive number a. By symmetry, the third vertex must be $(-1,-1)$. The distance between the vertex $(-1,-1)$ and the centroid $(1,1)$ is $2\sqrt{2}$, so the altitude of the triangle must be $\frac{3}{2} \cdot 2\sqrt{2} = 3\sqrt{2}$, which makes the side length of the triangle $s = \frac{2}{\sqrt{3}} \cdot 3\sqrt{2} = 2\sqrt{6}$. The required area is $\frac{\sqrt{3}}{4}s^2 = 6\sqrt{3}$. The requested value is $(6\sqrt{3})^2 = 108$. In fact, the vertices of the equilateral triangle are $(-1,-1)$, $(2+\sqrt{3}, 2-\sqrt{3})$, and $(2-\sqrt{3}, 2+\sqrt{3})$.

2018A

24. **Answer (D):** Because AB is $\frac{5}{6}$ of $AB + AC$, it follows from the Angle Bisector Theorem that DF is $\frac{5}{6}$ of DE, and BG is $\frac{5}{6}$ of BC. Because trapezoids $FDBG$ and $EDBC$ have the same height, the area of $FDBG$ is $\frac{5}{6}$ of the area of $EDBC$. Furthermore, the area of $\triangle ADE$ is $\frac{1}{4}$ of the area of $\triangle ABC$, so its area is 30, and the area of trapezoid $EDBC$ is $120-30 = 90$. Therefore the area of quadrilateral $FDBG$ is $\frac{5}{6} \cdot 90 = 75$.



Note: The figure (not drawn to scale) shows the situation in which $\angle ACB$ is acute. In this case $BC \approx 59.0$ and $\angle BAC \approx 151°$. It is also possible for $\angle ACB$ to be obtuse, with $BC \approx 41.5$ and $\angle BAC \approx 29°$. These values can be calculated using the Law of Cosines and the sine formula for area.