8

COMBINATIONS

2003A

21. (D) The numbers of the three types of cookies must have a sum of six. Possible sets of whole numbers whose sum is six are

$$0,0,6;\ \ 0,1,5;\ \ 0,2,4;\ \ 0,3,3;\ \ 1,1,4;\ \ 1,2,3;\quad \text{and } 2,2,2.$$

Every ordering of each of these sets determines a different assortment of cookies. There are 3 orders for each of the sets

$$0,0,6;\ \ 0,3,3;\ \ \text{and } 1,1,4.$$

There are 6 orders for each of the sets

$$0,1,5;\ \ 0,2,4;\ \ \text{and } 1,2,3.$$

There is only one order for $2,2,2$. Therefore the total number of assortments of six cookies is $3 \cdot 3 + 3 \cdot 6 + 1 = 28$.

OR

Construct eight slots, six to place the cookies in and two to divide the cookies by type. Let the number of chocolate chip cookies be the number of slots to the left of the first divider, the number of oatmeal cookies be the number of slots between the two dividers, and the number of peanut butter cookies be the number of slots to the right of the second divider. For example, $111 \mid 11 \mid 1$ represents three chocolate chip cookies, two oatmeal cookies, and one peanut butter cookie. There are $\binom{8}{2} = 28$ ways to place the two dividers, so there are 28 ways to select the six cookies.

2006A

21. **(E)** There are 9000 four-digit positive integers. For those without a 2 or 3, the first digit could be one of the seven numbers 1, 4, 5, 6, 7, 8, or 9, and each of the other digits could be one of the eight numbers 0, 1, 4, 5, 6, 7, 8, or 9. So there are

$$9000 - 7 \cdot 8 \cdot 8 \cdot 8 = 5416$$

four-digit numbers with at least one digit that is a 2 or a 3.

2008B 21. **Answer (C):** Let the women be seated first. The first woman may sit in any of the 10 chairs. Because men and women must alternate, the number of choices for the remaining women is 4, 3, 2, and 1. Thus the number of possible seating arrangements for the women is $10 \cdot 4! = 240$. Without loss of generality, suppose that a woman sits in chair 1. Then this woman's spouse must sit in chair 4 or chair 8. If he sits in chair 4 then the women sitting in chairs 7, 3, 9, and 5 must have their spouses sitting in chairs 10, 6, 2, and 8, respectively. If he sits in chair 8 then the women sitting in chairs 5, 9, 3, and 7 must have their spouses sitting in chairs 2, 6, 10, and 4, respectively. So for each possible seating arrangement for the women there are two arrangements for the men. Hence, there are $2 \cdot 240 = 480$ possible seating arrangements.

2013A 21. **Answer (D):** For $1 \le k \le 11$, the number of coins remaining in the chest before the k^{th} pirate takes a share is $\frac{12}{12-k}$ times the number remaining afterward. Thus if there are n coins left for the 12^{th} pirate to take, the number of coins originally in the chest is

$$\frac{12^{11} \cdot n}{11!} = \frac{2^{22} \cdot 3^{11} \cdot n}{2^8 \cdot 3^4 \cdot 5^2 \cdot 7 \cdot 11} = \frac{2^{14} \cdot 3^7 \cdot n}{5^2 \cdot 7 \cdot 11}.$$

The smallest value of n for which this is a positive integer is $5^2 \cdot 7 \cdot 11 = 1925$. In this case there are

$$2^{14} \cdot 3^7 \cdot \frac{11!}{(12-k)! \cdot 12^{k-1}}$$

coins left for the k^{th} pirate to take, and note that this amount is an integer for each k. Hence the 12^{th} pirate receives 1925 coins.

2010A

22. **Answer (A):** Three chords create a triangle if and only if they intersect pairwise inside the circle. Two chords intersect inside the circle if and only if their endpoints alternate in order around the circle. Therefore, if points A, B, C, D, E, and F are in order around the circle, then only the chords $\overline{AD}$, $\overline{BE}$, $\overline{CF}$ all intersect pairwise inside the circle. Thus every set of 6 points determines a unique triangle, and there are $\binom{8}{6} = 28$ such triangles.

2010B

22. **Answer (C):** If there were no restrictions on the number of candies per bag, then each piece of candy could be distributed in 3 ways. In this case there would be 3^7 ways to distribute the candy. However, this counts the cases where the red bag or blue bag is empty.

If the red bag remained empty then the candy could be distributed in 2^7 ways. The same is true for the blue bag. Both totals include the case in which all the candy is put into the white bag. Hence there are $2^7 + 2^7 - 1$ ways to distribute the candy such that either the red or blue bag is empty.

The number of ways to distribute the candy, subject to the given conditions, is $3^7 - (2^7 + 2^7 - 1) = 1932$.

2011A

22. **Answer (C):** If five distinct colors are used, then there are $\binom{6}{5} = 6$ different color choices possible. They may be arranged in $5! = 120$ ways on the pentagon, resulting in $120 \cdot 6 = 720$ colorings.

If four distinct colors are used, then there is one duplicated color, so there are $\binom{6}{4}\binom{4}{1} = 60$ different color choices possible. The duplicated color must appear on neighboring vertices. There are 5 neighbor choices and $3! = 6$ ways to color the remaining three vertices, resulting in a total of $60 \cdot 5 \cdot 6 = 1800$ colorings.

If three distinct colors are used, then there must be two duplicated colors, so there are $\binom{6}{3}\binom{3}{2} = 60$ different color choices possible. The non-duplicated color may appear in 5 locations. As before, a duplicated color must appear on neighboring vertices, so there are 2 ways left to color the remaining vertices. In this case there are $60 \cdot 5 \cdot 2 = 600$ colorings possible.

There are no colorings with two or fewer colors. The total number of colorings is $720 + 1800 + 600 = 3120$.

2012B

22. **Answer (A):** The sum of the first k positive integers is $\frac{k(k+1)}{2}$. Therefore the sum of the first k even integers is

$$2 + 4 + 6 + \cdots + 2k = 2(1 + 2 + 3 + \cdots + k) = 2 \cdot \frac{k(k+1)}{2} = k(k+1).$$

The sum of the first k odd integers is

$$(1 + 2 + 3 + \cdots + 2k) - (2 + 4 + 6 + \cdots + 2k) = \frac{2k(2k+1)}{2} - k(k+1) = k^2.$$

The given conditions imply that $m^2 - 212 = n(n+1)$, which may be rewritten as $n^2 + n + (212 - m^2) = 0$. The discriminant for n in this quadratic equation is $1 - 4(212 - m^2) = 4m^2 - 847$, and this must be the square of an odd integer. Let $p^2 = 4m^2 - 847$, and rearrange this equation so that $(2m+p)(2m-p) = 847$.

The only factor pairs for 847 are $847 \cdot 1$, $121 \cdot 7$, and $77 \cdot 11$. Equating these pairs to $2m + p$ and $2m - p$ yields $(m, p) = (212, 423)$, $(32, 57)$, and $(22, 33)$. Note that the corresponding values of n are found using $n = \frac{-1+p}{2}$, which yields 211, 28, and 16, respectively. The sum of the possible values of n is 255.

2016B

22. **Answer (A):** There must have been $10+10+1=21$ teams, and therefore there were $\binom{21}{3}=\frac{21\cdot 20\cdot 19}{6}=1330$ subsets $\{A,B,C\}$ of three teams. If such a subset does not satisfy the stated condition, then it consists of a team that beat both of the others. To count such subsets, note that there are 21 choices for the winning team and $\binom{10}{2}=45$ choices for the other two teams in the subset. This gives $21\cdot 45=945$ such subsets. The required answer is $1330-945=385$. To see that such a scenario is possible, arrange the teams in a circle, and let each team beat the 10 teams that follow it in clockwise order around the circle.

2008A

23. **Answer (B):** Let the two subsets be A and B. There are $\binom{5}{2}=10$ ways to choose the two elements common to A and B. There are then $2^3=8$ ways to assign the remaining three elements to A or B, so there are 80 ordered pairs (A,B) that meet the required conditions. However, the ordered pairs (A,B) and (B,A) represent the same pair $\{A,B\}$ of subsets, so the conditions can be met in $\frac{80}{2}=40$ ways.

2008B

23. **Answer (B):** Because the area of the border is half the area of the floor, the same is true of the painted rectangle. The painted rectangle measures $a-2$ by $b-2$ feet. Hence $ab=2(a-2)(b-2)$, from which $0=ab-4a-4b+8$. Add 8 to each side of the equation to produce

$$8=ab-4a-4b+16=(a-4)(b-4).$$

Because the only integer factorizations of 8 are

$$8=1\cdot 8=2\cdot 4=(-4)\cdot(-2)=(-8)\cdot(-1),$$

and because $b>a>0$, the only possible ordered pairs satisfying this equation for $(a-4,b-4)$ are $(1,8)$ and $(2,4)$. Hence (a,b) must be one of the two ordered pairs $(5,12)$, or $(6,8)$.

2010B

23. **Answer (D):** Let a_{ij} denote the entry in row i and column j. he given conditions imply that $a_{11} = 1$, $a_{33} = 9$, and $a_{22} = 4, 5$, or 6. If $a_{22} = 4$, then $\{a_{12}, a_{21}\} = \{2, 3\}$, and the sets $\{a_{31}, a_{32}\}$ and $\{a_{13}, a_{23}\}$ are complementary subsets of $\{5, 6, 7, 8\}$. There are $\binom{4}{2} = 6$ ways to choose $\{a_{31}, a_{32}\}$ and $\{a_{13}, a_{23}\}$, and only one way to order the entries. There are 2 ways to order $\{a_{12}, a_{21}\}$, so 12 arrays with $a_{22} = 4$ meet the given conditions. Similarly, the conditions are met by 12 arrays with $a_{22} = 6$. If $a_{22} = 5$, then $\{a_{12}, a_{13}, a_{23}\}$ and $\{a_{21}, a_{31}, a_{32}\}$ are complementary subsets of $\{2, 3, 4, 6, 7, 8\}$ subject to the conditions $a_{12} < 5$, $a_{21} < 5$, $a_{32} > 5$, and $a_{23} > 5$. Thus $\{a_{12}, a_{13}, a_{23}\} \neq \{2, 3, 4\}$ or $\{6, 7, 8\}$, so its elements can be chosen in $\binom{6}{3} - 2 = 18$ ways. Both the remaining entries and the ordering of all entries are then determined, so 18 arrays with $a_{22} = 5$ meet the given conditions.

Altogether, the conditions are met by $12 + 12 + 18 = 42$ arrays.

2012A

23. **Answer (B):** This situation can be modeled with a graph having these six people as vertices, in which two vertices are joined by an edge if and only if the corresponding people are internet friends. Let n be the number of friends each person has; then $1 \le n \le 4$. If $n = 1$, then the graph consists of three edges sharing no endpoints. There are 5 choices for Adam's friend and then 3 ways to partition the remaining 4 people into 2 pairs of friends, for a total of $5 \cdot 3 = 15$ possibilities. The case $n = 4$ is complementary, with non-friendship playing the role of friendship, so there are 15 possibilities in that case as well.

For $n = 2$, the graph must consist of cycles, and the only two choices are two triangles (3-cycles) and a hexagon (6-cycle). In the former case, there are $\binom{5}{2} = 10$ ways to choose two friends for Adam and that choice uniquely determines the triangles. In the latter case, every permutation of the six vertices determines a hexagon, but each hexagon is counted $6 \cdot 2 = 12$ times, because the hexagon can start at any vertex and be traversed in either direction. This gives $\frac{6!}{12} = 60$ hexagons, for a total of $10 + 60 = 70$ possibilities. The complementary case $n = 3$ provides 70 more. The total is therefore $15 + 15 + 70 + 70 = 170$.

2003A

24. (E) Let R1, . . ., R5 and B3, . . ., B6 denote the numbers on the red and blue cards, respectively. Note that R4 and R5 divide evenly into only B4 and B5, respectively. Thus the stack must be R4, B4, . . ., B5, R5, or the reverse. Since R2 divides evenly into only B4 and B6, we must have R4, B4, R2, B6, . . ., B5, R5, or the reverse. Since R3 divides evenly into only B3 and B6, the stack must be R4, B4, R2, B6, R3, B3, R1, B5, R5, or the reverse. In either case, the sum of the middle three cards is 12.

2012B

24. **Answer (E):** Adding the two equations gives

$$2a^2 + 2b^2 + 2c^2 - 2ab - 2bc - 2ac = 14,$$

so

$$(a - b)^2 + (b - c)^2 + (c - a)^2 = 14.$$

Note that there is a unique way to express 14 as the sum of perfect squares (up to permutations), namely, $14 = 3^2 + 2^2 + 1^2$. Because $a - b$, $b - c$, and $c - a$ are integers with their sum equal to 0 and $a \geq b \geq c$, it follows that $a - c = 3$ and either $a - b = 2$ and $b - c = 1$, or $a - b = 1$ and $b - c = 2$. Therefore either $(a, b, c) = (c + 3, c + 1, c)$ or $(a, b, c) = (c + 3, c + 2, c)$. Substituting the relations in the first case into the first given equation yields $2011 = a^2 - c^2 + ab - b^2 = (a - c)(a + c) + (a - b)b = 3(2c + 3) + 2(c + 1)$. Solving gives $(a, b, c) = (253, 251, 250)$. The second case does not yield an integer solution. Therefore $a = 253$.

2013A

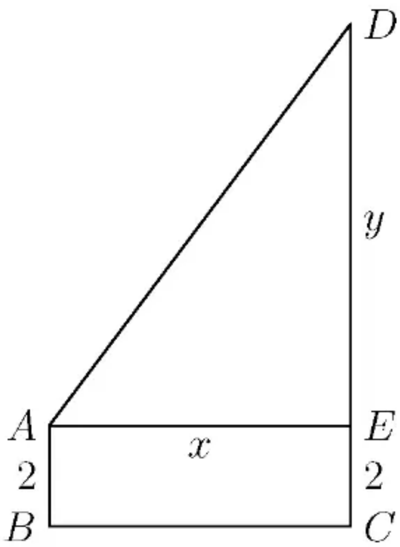
24. **Answer (E):** Call the players from Central A, B, and C, and call the players from Northern X, Y, and Z. Represent the schedule for each Central player by a string of length six consisting of two each of X, Y, and Z. There are $\binom{6}{2}\binom{4}{2}$ $= 90$ possible strings for player A. Assume without loss of generality that the string is $XXYYZZ$. Player B's schedule must be a string with no X's in the first two positions, no Y's in the next two, and no Z's in the last two. If B's string begins with a Y and a Z in either order, the next two letters must be an X and a Z, and the last two must be an X and a Y. Because each pair can be ordered in one of two ways, there are $2^3 = 8$ such strings. If B's string begins with YY or ZZ, it must be $YYZZXX$ or $ZZXXYY$, respectively. Hence there are 10 possible schedules for B for each of the 90 schedules for A, and C's schedule is then determined. The total number of possible schedules is 900.

2014B

24. **Answer (B):** The circular arrangement 14352 is bad because the sum 6 cannot be achieved with consecutive numbers, and the circular arrangement 23154 is bad because the sum 7 cannot be so achieved. It remains to show that these are the only bad arrangements. Given a circular arrangement, sums 1 through 5 can be achieved with a single number, and if the sum n can be achieved, then the sum $15 - n$ can be achieved using the complementary subset. Therefore an arrangement is not bad as long as sums 6 and 7 can be achieved. Suppose 6 cannot be achieved. Then 1 and 5 cannot be adjacent, so by a suitable rotation and/or reflection, the arrangement is $1bc5e$. Furthermore, $\{b, c\}$ cannot equal $\{2, 3\}$ because $1 + 2 + 3 = 6$; similarly $\{b, c\}$ cannot equal $\{2, 4\}$. It follows that $e = 2$, which then forces the arrangement to be 14352 in order to avoid consecutive 213. This arrangement is bad. Next suppose that 7 cannot be achieved. Then 2 and 5 cannot be adjacent, so again without loss of generality the arrangement is $2bc5e$. Reasoning as before, $\{b, c\}$ cannot equal $\{3, 4\}$ or $\{1, 4\}$, so $e = 4$, and then $b = 3$ and $c = 1$, to avoid consecutive 421; therefore the arrangement is 23154, which is also bad. Thus there are only two bad arrangements up to rotation and reflection.

2015A

24. **Answer (B):** In every such quadrilateral, $CD \geq AB$. Let E be the foot of the perpendicular from A to $\overline{CD}$; then $CE = 2$ and $AE = BC$. Let $x = AE$ and $y = DE$; then $AD = 2+y$. By the Pythagorean Theorem, $x^2+y^2 = (2+y)^2$, or $x^2 = 4+4y$. Therefore x is even, say $x = 2z$, and $z^2 = 1+y$. The perimeter of the quadrilateral is $x+2y+6 = 2z^2+2z+4$. Increasing positive integer values of z give the required quadrilaterals, with increasing perimeter. For $z = 31$ the perimeter is 1988, and for $z = 32$ the perimeter is 2116. Therefore there are 31 such quadrilaterals.



2003A

25. (B) Note that $n = 100q + r = q + r + 99q$. Hence $q + r$ is divisible by 11 if and only if n is divisible by 11. Since $10,000 \leq n \leq 99,999$, there are

$$\left\lfloor \frac{99999}{11} \right\rfloor - \left\lfloor \frac{9999}{11} \right\rfloor = 9090 - 909 = 8181$$

such numbers.

2005B 25. **(C)** Several pairs of numbers from 1 to 100 sum to 125. These pairs are $(25, 100)$, $(26, 99), \ldots, (62, 63)$. Set B can have at most one number from each of these $62-25+1 = 38$ pairs. In addition, B can contain all of the numbers $1, 2, \ldots, 24$ since these cannot be paired with any of the available numbers to sum to 125. So B has at most $38+24 = 62$ numbers. The set containing the first 62 positive integers, for example, is one of these maximum sets.

2012B 25. **Answer (D):** It may be assumed that $x \leq y \leq z$. Because there are six possible ways of permuting the triple (x, y, z), it follows that the set of all triples (x, y, z) with $0 \leq x \leq y \leq z \leq n$ is a region whose volume is $\frac{1}{6}$ of the volume of the cube $[0, n]^3$, that is $\frac{1}{6}n^3$. Let S be the set of triples meeting the required condition. For every $(x, y, z) \in S$ consider the translation $(x, y, z) \mapsto (x', y', z') = (x, y-1, z-2)$. Note that $y' = y - 1 > x = x'$ and $z' = z - 2 > y - 1 = y'$. Thus the image of S under this translation is equal to $\{(x', y', z') : 0 \leq x' < y' < z' \leq n-2\}$. Again by symmetry of the possible permutations of the triples (x', y', z'), the volume of this set is $\frac{1}{6}(n-2)^3$. Because $\frac{7^3}{9^3} = \frac{343}{729} < \frac{1}{2}$ and $\frac{8^3}{10^3} = \frac{512}{1000} > \frac{1}{2}$, the smallest possible value of n is 10.

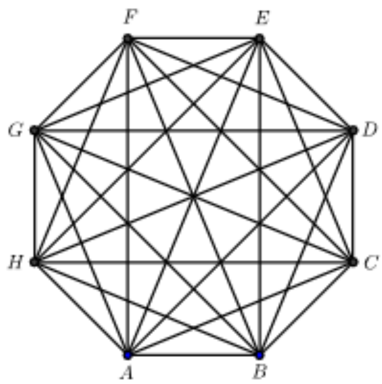
2013A

25. **Answer (A):** Label the octagon $ABCDEFGH$. There are 20 diagonals in all, 5 with endpoints at each vertex. The diagonals are of three types:

- Diagonals that skip over only one vertex, such as $\overline{AC}$ or $\overline{AG}$. These diagonals intersect with each of the five diagonals with endpoints at the skipped vertex.
- Diagonals that skip two vertices, such as $\overline{AD}$ or $\overline{AF}$. These diagonals intersect with four of the five diagonals that have endpoints at each of the two skipped vertices.
- Diagonals that cross to the opposite vertex, such as $\overline{AE}$. These diagonals intersect with three of the five diagonals that have endpoints at each of the three skipped vertices.

Therefore, from any given vertex, the diagonals will intersect other diagonals at $2\cdot5+2\cdot8+1\cdot9 = 35$ points. Counting from all 8 vertices, the total is $8\cdot35 = 280$ points.

Observe that, by symmetry, all four diagonals that cross to the opposite vertex intersect in the center of the octagon. This single intersection point has been counted 24 times, 3 from each of the 8 vertices. Further observe that at each of the vertices of the smallest internal octagon created by the diagonals, 3 diagonals intersect. For example, $\overline{AD}$ intersects with $\overline{CH}$ on $\overline{BF}$. These 8 intersection points have each been counted 12 times, 2 from each of the 6 affected vertices.The remaining intersection points each involve only two diagonals and each has been counted 4 times, once from each endpoint. These number $\frac{280-24-8\cdot12}{4} = 40$. There are therefore $1 + 8 + 40 = 49$ distinct intersection points in the interior of the octagon.



2017A

25. **Answer (A):** Recall the divisibility test for 11: A three-digit number $\underline{a}\,\underline{b}\,\underline{c}$ is divisible by 11 if and only if $a - b + c$ is divisible by 11. The smallest and largest three-digit multiples of 11 are, respectively, $110 = 10 \cdot 11$ and $990 = 90 \cdot 11$, so the number of three-digit multiples of 11 is $90 - 10 + 1 = 81$. They may be grouped as follows:

- There are 9 multiples of 11 that have the form $\underline{a}\,\underline{a}\,\underline{0}$ for $1 \le a \le 9$. They can each be permuted to form a total of 2 three-digit integers. In each case $\underline{a}\,\underline{a}\,\underline{0}$ is a multiple of 11 and $\underline{a}\,\underline{0}\,\underline{a}$ is not, so these 9 multiples of 11 give 18 integers with the required property.

- There are 8 multiples of 11 that have the form $\underline{a}\,\underline{b}\,\underline{a}$, namely 121, 242, 363, 484, 616, 737, 858, and 979. They can each be permuted to form a total of 3 three-digit integers. In each case $\underline{a}\,\underline{b}\,\underline{a}$ is a multiple of 11, but neither $\underline{a}\,\underline{a}\,\underline{b}$ nor $\underline{b}\,\underline{a}\,\underline{a}$ is, so these 8 multiples of 11 give 24 integers with the required property.

- If a three-digit multiple of 11 has distinct digits and one digit is 0, it must have the form $\underline{a}\,\underline{0}\,\underline{c}$ with $a + c = 11$. There are 8 such integers, namely 209, 308, 407, $\ldots$, 902. They can each be permuted to form a total of 4 three-digit integers, but these 8 multiples of 11 give only 4 distinct sets of permutations, leading to $4 \cdot 4 = 16$ integers with the required property.

- The remaining $81 - (9 + 8 + 8) = 56$ three-digit multiples of 11 all have the form $\underline{a}\,\underline{b}\,\underline{c}$, where a, b, and c are distinct nonzero digits. They can each be permuted to form a total of 6 three-digit integers, and in each case both $\underline{a}\,\underline{b}\,\underline{c}$ and $\underline{c}\,\underline{b}\,\underline{a}$—and only these—are multiples of 11. Therefore these 56 multiples of 11 give only 28 distinct sets of permutations, leading to $28 \cdot 6 = 168$ integers with the required property.

The total number of integers with the required property is $18 + 24 + 16 + 168 = 226$.