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ARITHMETIC

2004B 6. **(C)** Note that for $m < n$ we have

$$m! \cdot n! = (m!)^2 \cdot (m+1) \cdot (m+2) \cdot \cdots \cdot n.$$

Therefore $m! \cdot n!$ is a perfect square if and only if

$$(m+1) \cdot (m+2) \cdot \cdots \cdot n$$

is a perfect square. For the five answer choices, that quantity is

$$99, \ 99 \cdot 100, \ 100, \ 100 \cdot 101, \ \text{and} \ \ 101,$$

and of those only 100 is a perfect square. Therefore the answer is $99! \cdot 100!$.

2003A 6. (C) For example, $-1 \heartsuit 0 = |-1-0| = 1 \neq -1$. All the other statements are true:

(A) $x \heartsuit y = |x-y| = |-(y-x)| = |y-x| = y \heartsuit x$ for all x and y.
(B) $2(x \heartsuit y) = 2|x-y| = |2x-2y| = (2x) \heartsuit (2y)$ for all x and y.
(D) $x \heartsuit x = |x-x| = 0$ for all x.
(E) $x \heartsuit y = |x-y| > 0$ if $x \neq y$.

2010A

6. **Answer (C):** Note that $\spadesuit(2,2) = 2 - \frac{1}{2} = \frac{3}{2}$. Therefore

$$\spadesuit(2, \spadesuit(2,2)) = \spadesuit\left(2, \frac{3}{2}\right) = 2 - \frac{2}{3} = \frac{4}{3}.$$

2003B

7. (B) The first three values in the sum are 1, the next five are 2, the next seven are 3, and the final one is 4 for a total of

$$3 \cdot 1 + 5 \cdot 2 + 7 \cdot 3 + 1 \cdot 4 = 38.$$

2006B

7. **(A)** We have

$$\sqrt{\frac{x}{1 - \frac{x-1}{x}}} = \sqrt{\frac{x}{\frac{x-x+1}{x}}} = \sqrt{\frac{x}{\frac{1}{x}}} = \sqrt{x^2} = |x|.$$

When $x < 0$, the given expression is equivalent to $-x$.

2009B

7. **Answer (C):** The three operations can be performed in any of $3! = 6$ orders. However, if the addition is performed either first or last, then multiplying in either order produces the same result. Thus at most four distinct values can be obtained. It is easily checked that the values of the four expressions

$$\begin{aligned}(2 \times 3) + (4 \times 5) &= 26,\\ ((2 \times 3 + 4) \times 5) &= 50,\\ 2 \times (3 + (4 \times 5)) &= 46,\\ 2 \times (3 + 4) \times 5 &= 70\end{aligned}$$

are in fact all distinct.

2013A

8. **Answer (C):** Factoring 2^{2012} from each of the terms and simplifying gives

$$\frac{2^{2012}(2^2+1)}{2^{2012}(2^2-1)} = \frac{4+1}{4-1} = \frac{5}{3}.$$

2008A

7. **Answer (E):** First note that

$$\frac{\left(3^{2008}\right)^2 - \left(3^{2006}\right)^2}{\left(3^{2007}\right)^2 - \left(3^{2005}\right)^2} = \frac{9^{2008} - 9^{2006}}{9^{2007} - 9^{2005}}.$$

Factoring 9^{2005} from each of the terms on the right side produces

$$\frac{9^{2008} - 9^{2006}}{9^{2007} - 9^{2005}} = \frac{9^{2005}\cdot 9^3 - 9^{2005}\cdot 9^1}{9^{2005}\cdot 9^2 - 9^{2005}\cdot 1} = \frac{9^{2005}}{9^{2005}}\cdot\frac{9^3-9}{9^2-1} = 9\cdot\frac{9^2-1}{9^2-1} = 9.$$

2015B

7. **Answer (A):**

$$((1\diamond 2)\diamond 3) - (1\diamond(2\diamond 3)) = \left(\left(1-\frac{1}{2}\right)-\frac{1}{3}\right) - \left(1-\left(2-\frac{1}{3}\right)^{-1}\right)$$
$$= \frac{1}{6} - \left(1-\frac{3}{5}\right) = \frac{1}{6} - \frac{2}{5} = -\frac{7}{30}$$

2016B

8. **Answer (A):** Positive even powers of numbers ending in 5 end in 25. The tens digit of the difference is the tens digit of $25 - 17 = 08$, or 0.

2014A

8. **Answer (D):** Note that $\frac{n!(n+1)!}{2} = (n!)^2 \cdot \frac{n+1}{2}$, which is a perfect square if and only if $\frac{n+1}{2}$ is a perfect square. Only choice D satisfies this condition.

2003A 9. (A) We have

$$\begin{aligned}
\sqrt[3]{x\sqrt[3]{x\sqrt[3]{x\sqrt{x}}}} &= (x(x(x \cdot x^{\frac{1}{2}})^{\frac{1}{3}})^{\frac{1}{3}})^{\frac{1}{3}} \\
&= (x(x(x^{\frac{3}{2}})^{\frac{1}{3}})^{\frac{1}{3}})^{\frac{1}{3}} \\
&= (x(x \cdot x^{\frac{1}{2}})^{\frac{1}{3}})^{\frac{1}{3}} \\
&= (x(x^{\frac{3}{2}})^{\frac{1}{3}})^{\frac{1}{3}} = (x \cdot x^{\frac{1}{2}})^{\frac{1}{3}} = (x^{\frac{3}{2}})^{\frac{1}{3}} = x^{\frac{1}{2}} = \sqrt{x}.
\end{aligned}$$