MAA AMC
American Mathematics Competitions

AMC 10 A

DO NOT OPEN UNTIL THURSDAY, January 30, 2020

****Administration on an earlier date will disqualify your school's results.****

- All the information needed to administer this competition is contained in the AMC 10/12 Teacher's Manual. PLEASE READ THE MANUAL BEFORE THURSDAY, JANUARY 30, 2020.
- Answer sheets must be returned to the MAA AMC office within 24 hours of the competition administration. Use an overnight or 2-day shipping service, with a tracking number, to guarantee timely arrival of these answer sheets. FedEx, UPS, or USPS overnight are strongly recommended.
- The 38th annual American Invitational Mathematics Exam will be held on Wednesday, March 11, 2020, with an alternate date on Thursday, March 19, 2020. It is a 15-question, 3-hour, integer-answer competition. Students who achieve a high score on the AMC 10 will be invited to participate. Top-scoring students on the AMC 10/12 and AIME will be selected to take the USA (Junior) Mathematical Olympiad. The USA(J)MO will be given on Tuesday and Wednesday, April 14 and 15, 2020.
- The publication, reproduction, or communication of the problems or solutions of this competition during the period when students are eligible to participate seriously jeopardizes the integrity of the results. Dissemination via phone, email, or digital media of any type during this period is a violation of the competition rules.

MAA AMC
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21st Annual

AMC 10 A

Thursday, January 30, 2020

INSTRUCTIONS

1. DO NOT OPEN THIS BOOKLET UNTIL YOUR COMPETITION MANAGER TELLS YOU TO BEGIN.
2. This is a 25-question multiple-choice competition. For each question, only one answer choice is correct.
3. Mark your answer to each problem on the answer sheet with a #2 pencil. Check blackened answers for accuracy and erase errors completely. Only answers that are properly marked on the answer sheet will be scored.
4. SCORING: You will receive 6 points for each correct answer, 1.5 points for each problem left unanswered, and 0 points for each incorrect answer.
5. Only blank scratch paper, blank graph paper, rulers, compasses, protractors, and erasers are allowed as aids. No calculators, smartwatches, phones, or computing devices are allowed. No problems on the competition will require the use of a calculator.
6. Figures are not necessarily drawn to scale.
7. Before beginning the competition, your competition manager will ask you to record your name on the answer sheet.
8. You will have 75 minutes to complete the competition once your competition manager tells you to begin.
9. When you finish the competition, sign your name in the space provided on the answer sheet and complete the demographic information question on the back of the answer sheet.

The MAA AMC Office reserves the right to disqualify scores from a school if it determines that the rules or the required security procedures were not followed.

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Students who score well on this AMC 10 will be invited to take the 38th annual American Invitational Mathematics Examination (AIME) on Wednesday, March 11, 2020, or Thursday, March 19, 2020. More details about the AIME are on the back page of this test booklet.

1. What value of x satisfies

$$x - \frac{3}{4} = \frac{5}{12} - \frac{1}{3}?$$

(A) $-\frac{2}{3}$ **(B)** $\frac{7}{36}$ **(C)** $\frac{7}{12}$ **(D)** $\frac{2}{3}$ **(E)** $\frac{5}{6}$

2. The numbers 3, 5, 7, a, and b have an average (arithmetic mean) of 15. What is the average of a and b?

(A) 0 **(B)** 15 **(C)** 30 **(D)** 45 **(E)** 60

3. Assuming $a \neq 3$, $b \neq 4$, and $c \neq 5$, what is the value in simplest form of the following expression?

$$\frac{a-3}{5-c} \cdot \frac{b-4}{3-a} \cdot \frac{c-5}{4-b}$$

(A) -1 **(B)** 1 **(C)** $\frac{abc}{60}$ **(D)** $\frac{1}{abc} - \frac{1}{60}$ **(E)** $\frac{1}{60} - \frac{1}{abc}$

4. A driver travels for 2 hours at 60 miles per hour, during which her car gets 30 miles per gallon of gasoline. She is paid \$0.50 per mile, and her only expense is gasoline at \$2.00 per gallon. What is her net rate of pay, in dollars per hour, after this expense?

(A) 20 **(B)** 22 **(C)** 24 **(D)** 25 **(E)** 26

5. What is the sum of all real numbers x for which $|x^2 - 12x + 34| = 2$?

(A) 12 **(B)** 15 **(C)** 18 **(D)** 21 **(E)** 25

6. How many 4-digit positive integers (that is, integers between 1000 and 9999, inclusive) having only even digits are divisible by 5?

(A) 80 **(B)** 100 **(C)** 125 **(D)** 200 **(E)** 500

7. The 25 integers from -10 to 14, inclusive, can be arranged to form a 5-by-5 square in which the sum of the numbers in each row, the sum of the numbers in each column, and the sum of the numbers along each of the main diagonals are all the same. What is the value of this common sum?

(A) 2 **(B)** 5 **(C)** 10 **(D)** 25 **(E)** 50

8. What is the value of

$$1 + 2 + 3 - 4 + 5 + 6 + 7 - 8 + \cdots + 197 + 198 + 199 - 200?$$

(A) 9,800 **(B)** 9,900 **(C)** 10,000 **(D)** 10,100 **(E)** 10,200

9. A single bench section at a school event can hold either 7 adults or 11 children. When N bench sections are connected end to end, an equal number of adults and children seated together will occupy all the bench space. What is the least possible positive integer value of N?

(A) 9 **(B)** 18 **(C)** 27 **(D)** 36 **(E)** 77

10. Seven cubes, whose volumes are 1, 8, 27, 64, 125, 216, and 343 cubic units, are stacked vertically to form a tower in which the volumes of the cubes decrease from bottom to top. Except for the bottom cube, the bottom face of each cube lies completely on top of the cube below it. What is the total surface area of the tower (including the bottom) in square units?

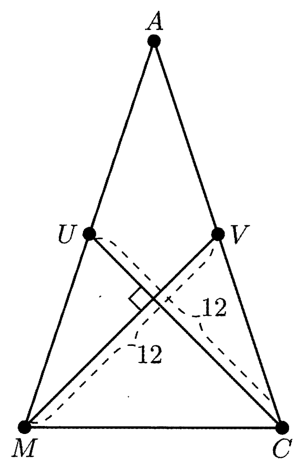
(A) 644 **(B)** 658 **(C)** 664 **(D)** 720 **(E)** 749

11. What is the median of the following list of 4040 numbers?

$$1, 2, 3, \ldots, 2020, 1^2, 2^2, 3^2, \ldots, 2020^2$$

(A) 1974.5 **(B)** 1975.5 **(C)** 1976.5 **(D)** 1977.5 **(E)** 1978.5

12. Triangle AMC is isosceles with $AM = AC$. Medians $\overline{MV}$ and $\overline{CU}$ are perpendicular to each other, and $MV = CU = 12$. What is the area of $\triangle AMC$?



(A) 48 **(B)** 72 **(C)** 96 **(D)** 144 **(E)** 192

13. A frog sitting at the point $(1, 2)$ begins a sequence of jumps, where each jump is parallel to one of the coordinate axes and has length 1, and the direction of each jump (up, down, right, or left) is chosen independently at random. The sequence ends when the frog reaches a side of the square with vertices $(0, 0)$, $(0, 4)$, $(4, 4)$, and $(4, 0)$. What is the probability that the sequence of jumps ends on a vertical side of the square?

(A) $\frac{1}{2}$ **(B)** $\frac{5}{8}$ **(C)** $\frac{2}{3}$ **(D)** $\frac{3}{4}$ **(E)** $\frac{7}{8}$

14. Real numbers x and y satisfy $x + y = 4$ and $x \cdot y = -2$. What is the value of

$$x + \frac{x^3}{y^2} + \frac{y^3}{x^2} + y?$$

(A) 360 **(B)** 400 **(C)** 420 **(D)** 440 **(E)** 480

15. A positive integer divisor of $12!$ is chosen at random. The probability that the divisor chosen is a perfect square can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. What is $m+n$?

(A) 3 **(B)** 5 **(C)** 12 **(D)** 18 **(E)** 23

16. A point is chosen at random within the square in the coordinate plane whose vertices are $(0, 0)$, $(2020, 0)$, $(2020, 2020)$, and $(0, 2020)$. The probability that the point lies within d units of a lattice point is $\frac{1}{2}$. (A point (x, y) is a lattice point if x and y are both integers.) What is d to the nearest tenth?

(A) 0.3 **(B)** 0.4 **(C)** 0.5 **(D)** 0.6 **(E)** 0.7

17. Define

$$P(x) = \left(x - 1^2\right)\left(x - 2^2\right)\cdots\left(x - 100^2\right).$$

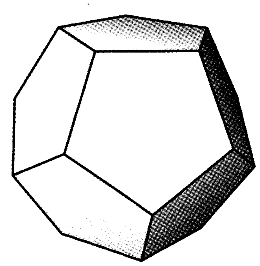
How many integers n are there such that $P(n) \le 0$?

(A) 4900 **(B)** 4950 **(C)** 5000 **(D)** 5050 **(E)** 5100

18. Let (a, b, c, d) be an ordered quadruple of not necessarily distinct integers, each one of them in the set $\{0, 1, 2, 3\}$. For how many such quadruples is it true that $a \cdot d - b \cdot c$ is odd? (For example, $(0, 3, 1, 1)$ is one such quadruple, because $0 \cdot 1 - 3 \cdot 1 = -3$ is odd.)

(A) 48 **(B)** 64 **(C)** 96 **(D)** 128 **(E)** 192

19. As shown in the figure below, a regular dodecahedron (the polyhedron consisting of 12 congruent regular pentagonal faces) floats in space with two horizontal faces. Note that there is a ring of five slanted faces adjacent to the top face, and a ring of five slanted faces adjacent to the bottom face. How many ways are there to move from the top face to the bottom face via a sequence of adjacent faces so that each face is visited at most once and moves are not permitted from the bottom ring to the top ring?



(A) 125 **(B)** 250 **(C)** 405 **(D)** 640 **(E)** 810

20. Quadrilateral $ABCD$ satisfies $\angle ABC = \angle ACD = 90°$, $AC = 20$, and $CD = 30$. Diagonals $\overline{AC}$ and $\overline{BD}$ intersect at point E, and $AE = 5$. What is the area of quadrilateral $ABCD$?

(A) 330 **(B)** 340 **(C)** 350 **(D)** 360 **(E)** 370

21. There exists a unique strictly increasing sequence of nonnegative integers $a_1 < a_2 < \cdots < a_k$ such that

$$\frac{2^{289}+1}{2^{17}+1} = 2^{a_1} + 2^{a_2} + \cdots + 2^{a_k}.$$

What is k?

(A) 117 **(B)** 136 **(C)** 137 **(D)** 273 **(E)** 306

22. For how many positive integers $n \le 1000$ is

$$\left\lfloor \frac{998}{n} \right\rfloor + \left\lfloor \frac{999}{n} \right\rfloor + \left\lfloor \frac{1000}{n} \right\rfloor$$

not divisible by 3? (Recall that $\lfloor x \rfloor$ is the greatest integer less than or equal to x.)

(A) 22 **(B)** 23 **(C)** 24 **(D)** 25 **(E)** 26

23. Let T be the triangle in the coordinate plane with vertices $(0,0)$, $(4,0)$, and $(0,3)$. Consider the following five isometries (rigid transformations) of the plane: rotations of $90°$, $180°$, and $270°$ counterclockwise around the origin, reflection across the x-axis, and reflection across the y-axis. How many of the 125 sequences of three of these transformations (not necessarily distinct) will return T to its original position? (For example, a $180°$ rotation, followed by a reflection across the x-axis, followed by a reflection across the y-axis will return T to its original position, but a $90°$ rotation, followed by a reflection across the x-axis, followed by another reflection across the x-axis will not return T to its original position.)

(A) 12 **(B)** 15 **(C)** 17 **(D)** 20 **(E)** 25

24. Let n be the least positive integer greater than 1000 for which

$$\gcd(63, n+120) = 21 \quad \text{and} \quad \gcd(n+63, 120) = 60.$$

What is the sum of the digits of n?

(A) 12 **(B)** 15 **(C)** 18 **(D)** 21 **(E)** 24

25. Jason rolls three fair standard six-sided dice. Then he looks at the rolls and chooses a subset of the dice (possibly empty, possibly all three dice) to reroll. After rerolling, he wins if and only if the sum of the numbers face up on the three dice is exactly 7. Jason always plays to optimize his chances of winning. What is the probability that he chooses to reroll exactly two of the dice?

(A) $\frac{7}{36}$ **(B)** $\frac{5}{24}$ **(C)** $\frac{2}{9}$ **(D)** $\frac{17}{72}$ **(E)** $\frac{1}{4}$